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Polynomial Interpolation to Data on Flats in $\mathbb{R}^d$

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Stimulated by recent work of Hakopian and Sahakian, polynomial interpolation to data at all the s-dimensional intersections of an arbitrary sequence of hyperplanes in $\mathbb{R}^d$ is considered, and reduced, by the adjunction of an additional s hyperplanes in general position with respect to the given sequence, to the case $s=0$ solved much earlier by two of the present authors. In particular, interpolation is from the very same polynomial spaces already used earlier. The difficult question of multiplicity and corresponding matching of derivative information is completely solved, with the number of independent derivative conditions at an intersection exactly equal to that intersection's multiplicity. Also, the consistency requirements placed on the data are minimal in the sense that they need to be checked only at the finitely many 0-dimensional intersections of the hyperplanes involved. The arguments used provide, incidentally, further insights into the two polynomial spaces, $\mathscr{P}(\Xi)$ and $\mathscr{D}(\Xi)$, of basic interest in box spline theory.

Key Words: polynomial interpolation; multivariate; linear manifold; box spline theory.

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1. INTRODUCTION

When going from univariate to multivariate polynomial interpolation, one can follow the point of view of [HS2] that a point in $\mathbb{R} = \mathbb{R}^1$ corresponds to a hyperplane in $\mathbb{R}^d$ and, correspondingly, attempt to interpolate to information given on *hyperplanes*. It is one message of the present paper that the efficient way to solve such interpolation problems is nevertheless via an equivalent problem of interpolation to (derived) information given at *points*. The relevant pointset is obtained as the set of all 0-dimensional intersections of hyperplanes from the set that comprises the given hyperplanes and $d-1$ additional hyperplanes in general position. Interpolation at such pointsets was studied already in [DR].

In [DR, BDR], we considered interpolation problems in which two polynomial spaces that emerge from box spline theory, $\mathscr{P}(\Xi)$ and $\mathscr{D}(\Xi)$, are involved. Usually, one of these spaces is used to determine the interpolation conditions and the interpolant is sought from the other. In the above-cited papers, the interpolation conditions are always of Lagrange type or Hermite type, i.e., we interpolate function values and "consecutive" derivative values at these points. Motivated by the recent interesting paper [HS2] of Hakopian and Sahakian, we study in this paper polynomial interpolation to data given on certain linear manifolds or **flats** (for short) in $\mathbb{R}^d$, all of the same dimension s, and this dimension is held fixed throughout. The case $s=0$ reduces to the interpolation problem considered in [DR].

The collection of s-dimensional flats involved is denoted by $\mathscr{M}_s(\mathbb{H})$; it consists of all s-dimensional intersections of hyperplanes taken from a given sequence $\mathbb{H}$.

Generically, each $M \in \mathscr{M}_s(\mathbb{H})$ is the intersection of exactly $d-s$ hyperplanes, and we call this situation (whether generic or not) the *simple case*, and expect, in this case, to match values given at all the flats in $\mathscr{M}_s(\mathbb{H})$, i.e., Lagrange type interpolation. The examples shown as Cases 1.1–1.3 in Section 2 are of this type.

In the contrary case (see, e.g., the examples shown as Cases 2.1–2.4 in Section 2), some $M \in \mathscr{M}_s(\mathbb{H})$ is the intersection of more than $d-s$ hyperplanes, and, correspondingly, we would expect to match at such M also some "successive" derivatives, leading to Hermite type interpolation. However, in contrast to [HS2], we would not demand the matching of all derivatives up to a certain order. Rather, in a ready generalization of the approach introduced in [DR], we impose Hermite conditions that can be shown to be exactly those satisfied, in a suitable limiting process, by a Lagrange interpolant to data taken from a smooth function. In this way, the resulting Hermite interpolation is osculatory or "repeated" interpolation in the classical sense.

Specification of data on flats of dimension $s>0$ raises the question of *consistency*: Since distinct $M, M' \in \mathscr{M}_s(\mathbb{H})$ may well have a nontrivial intersection, there is the possibility that the information supplied on M and M' is contradictory on $M \cap M'$. In the Lagrange case, this is simply a question of having the two polynomials, specified on M and M', respectively, coincide on $M \cap M'$. In the Hermite case, matters are a bit more subtle. Fortunately, in contrast to [HS2], we are able to reduce all such consistency questions to checking consistency at a certain *finite* set (namely the set $\mathscr{M}_0(\mathbb{H}_s)$; see below).

Given that, even in the simple case, the cardinality of $\mathscr{M}_s(\mathbb{H})$ is not just a function of $\#\mathbb{H}$, we must also specify a suitable $\mathbb{H}$-dependent polynomial space from which to interpolate. This was already recognized in [DR] where the case $s=0$ of our current problem was settled. In fact, our approach reduces the general problem to the special case treated in [DR]: we extract from the given data a discrete (finite) set of Hermite type interpolation conditions, and choose the interpolant from a polynomial space $\mathscr{P}=\mathscr{P}_s(\mathbb{H})$ which depends only on the sequence $\mathbb{H}$ and the number s. We will not be able to match the information given at all the $M \in \mathscr{M}_s(\mathbb{H})$ by some $p \in \mathscr{P}$ unless the information is **compatible with** $\mathscr{P}$, i.e., unless the datum specified on a given $M \in \mathscr{M}_s(\mathbb{H})$ is taken from some element of $\mathscr{P}$ (with that element, offhand, different from datum to datum).

With this, our main result, Theorem 7.19, states that, *for an arbitrary finite sequence* $\mathbb{H}$ *of hyperplanes and arbitrary consistent and* $\mathscr{P}$*-compatible data, there is exactly one element* $p \in \mathscr{P}$ *that matches these data.*

Our proof of this result is quite technical, and provides, perhaps unexpected, insights into the structure of the spaces $\mathscr{D}(\Xi)$ which play such a central role in box spline theory (see, e.g., [BDR]). These insights are contained in Theorem 7.7 which is a consequence of Theorem 7.9, and in Theorem 7.13. It is our hope that these insights will also find use elsewhere.

The paper is laid out as follows. All of our functions hereafter are complex-valued and defined on $\mathbb{R}^d$. In order to simplify the presentation, we consider first (in Section 5) the simple case, i.e., the Lagrange type interpolation that is analysed here. This simplifies almost all aspects of the analysis: the description of the interpolation conditions, the compatibility requirements on the interpolation conditions, and the solution we provide to the interpolation problem. In contrast, the polynomial space that is to supply the interpolant is the same for the Lagrange and non-Lagrange problems, hence we need first to introduce and discuss that space, as we do in (Section 3 and) Section 4.

In Section 2, we illustrate our interpolation problems by treating the case of three lines in 2-space, and we outline, in Section 3, for the simple case in some detail the basic idea of our construction and proofs.

The general interpolation problem is analysed in Section 7. Some simple facts, concerning D-invariant polynomial subspaces, especially polynomial ideals and their polynomial kernels, needed there are discussed in Section 6.

The results in this paper were obtained in 1991, in reaction to [HS2]. We delayed publication initially so as not to precede publication of [HS2] and later because we wanted to follow [HS2] and include also a treatment of interpolation to information "at infinity". However, our results in that regard are still not complete while, at this time, there are signs of interest in [HS2] and we feel that, in any case, the present paper is already substantial enough.

2. AN EXAMPLE

We are given three (straight) lines in $\mathbb{R}^2$ and, on each line, a "univariate" quadratic polynomial (i.e., the restriction of some quadratic polynomial to that line), and seek to extend this information to a quadratic polynomial on all of $\mathbb{R}^2$. Of course, we assume that the data are consistent, but the precise meaning of this depends on the specific circumstances, as discussed below. In any event, we proceed by choosing an additional line, in general position, and then constructing the interpolant from point data, derived from the given data, at all the intersections of pairs of lines (see Fig. 1).

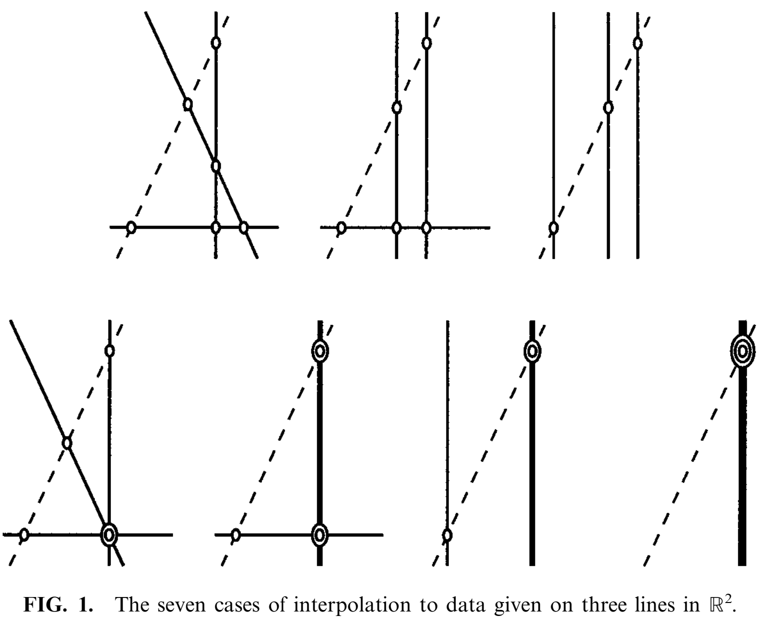

FIG. 1. The seven cases of interpolation to data given on three lines in $\mathbb{R}^2$.

Case 1.1 (*General Position*). The three given lines are in general position, meaning that any two have exactly one point in common, and this point does not lie on the third line.

In this case, a fourth line in general position will intersect each of the given lines at a point not also on another line, thus giving us a 6-point set that is well known (see, e.g., [DR]) to permit unique interpolation from the space Π_2 of bivariate quadratic polynomials to arbitrary data. In particular, let p be the interpolant from Π_2 to the data derived from the given quadratic polynomials. This requires consistency, in the sense that, at the point common to two given lines, the corresponding polynomials prescribed on these two lines have to agree. Then, on each of the given lines, p reduces to a "univariate" quadratic polynomial. That quadratic polynomial agrees with the original "univariate" quadratic polynomial given on the line, since they agree at three points on that line.

Case 1.2 (*Two* (*But Not Three*) *Lines Are Parallel* (*But Not Coincident*)). This is a limiting case of the Case 1.1, reached by rotating one of the lines.

Now our procedure produces only 5 points of intersection, hence interpolation from Π_2 would be underdetermined. In this situation, [HS2] derive, in effect, information at the point at infinity at which the two parallel lines intersect. We prefer to interpolate instead from a certain 5-dimensional linear subspace $\mathscr{P}(\Xi)$ of Π_2. Precisely, $\Xi = [\xi_1, ..., \xi_4]$ is a matrix whose jth column contains a (nontrivial) vector perpendicular to the jth line, $j = 1, ..., 4$, and $\mathscr{P}(\Xi)$ is the linear span of all functions of the form $x \mapsto \prod_{j \in J} (\xi_j \cdot x)$, with J such that $\{\xi_i : i \notin J\}$ spans $\mathbb{R}^2$.

In Case 1.1, $\mathscr{P}(\Xi) = \Pi_2$. In the present case, however, all elements of $\mathscr{P}(\Xi)$ are necessarily linear in the direction of the parallel lines (which implies that $\mathscr{P}(\Xi)$ is obtained from Π_2 by the imposition of one linear constraint). There is a unique interpolant from $\mathscr{P}(\Xi)$ to the derived data at the five points, provided that the data are *compatible with* $\mathscr{P}(\Xi)$, i.e., provided the polynomials given on the two parallel lines are actually linear (this is a special case of the general result in [DR], to be used in the sequel). With this proviso, the restriction of the interpolating polynomial to a given line agrees with the given polynomial there since it matches it there at as many points as are needed, given the degrees involved, to conclude agreement on the entire line from agreement at those points.

Case 1.3 (*Three Parallel Lines*). In this case, $\mathscr{P}(\Xi)$ (defined as in Case 1.2) reduces to the 3-dimensional space of all quadratic polynomials constant in the direction of those three lines. Compatibility of the data now means that they must be constant on each of those three lines, while consistency is vacuous here (since the lines have empty intersection). With that, existence of exactly one element of $\mathscr{P}(\Xi)$ matching such data is evident.

The three cases covered so far are examples of what we call the *simple case*; it leads to an interpolation problem of Lagrange type and the reader only interested in this case may safely skip the rest of this section.

In the *nonsimple*, or *general*, case, one has to deal with interpolation to derivative information as well, derived from the given information (as in Case 2.1 below) and/or given explicitly (as in Cases 2.2–2.4 below).

Case 2.1 (*The Three Lines Intersect at a Common Point*, *But No Two Lines Coincide*). This is a limiting case of Case 1.1, reached by translating (but not rotating) one of the lines. In particular, $\mathscr{P}(\Xi)$ does not change, i.e., it is still Π_2 for this case. Having the function specified on a straight line means, of course, that we have also specified, on that line, any derivative of any order in the direction of that line. Since, on $\mathbb{R}^2$, it takes just two directional derivatives at a point (in nonparallel directions) to specify every directional derivative at that point, consistency now requires that the three directional derivatives specified by the data at the point common to all three lines be consistent. That being understood, the rest is as before, with the slight complication that the common point is a triple point for interpolation from Π_2, and is a double point when arguing that the interpolant from Π_2 agrees with the given data on each of the three given lines.

Case 2.2 (*Two Lines Are Coincident*, *the Third Not Parallel*). This is a limiting case of Case 1.2, as we translate (but not rotate) the given lines suitably, hence $\mathscr{P}(\Xi)$ is the same as in Case 1.2. (It is also a limiting case of Case 2.1, as we rotate one of the lines suitably.)

We now assume given on the double line also the derivative normal to that line (necessarily a linear polynomial along that line even if we only knew that it was the normal derivative of some element of Π_2), hence know on that line any directional derivative. The only change from the preceding case is that some points now become double points and that we must (and can) also verify that the given normal derivative is matched on the entire double line.

Case 2.3 (*Three Parallel Lines*, *Two Coincident*). This is a limiting case of Case 1.3, as we translate (but not rotate) one of the given lines suitably, hence $\mathscr{P}(\Xi)$ is the same as in Case 1.3. (It is also a limiting case of Case 2.2, as we rotate one of the lines suitably.)

We now assume given on the double line also the derivative normal to that line, and compatibility requires (as in Case 1.3) that all data, including this normal derivative, be constant, making it easy to verify the existence of exactly one element in $\mathscr{P}(\Xi)$ that matches the given data.

Case 2.4 (*Three Coincident Lines*). This is a limiting case of Case 1.3 (and Case 2.3) as we translate (but not rotate) the given lines suitably, hence $\mathscr{P}(\Xi)$ is the same as in Cases 1.3 and 2.3. (It is also a limiting case of Case 2.2, as we rotate one of the lines suitably.)

In this case, assuming the data compatible with this, i.e., constant along the triple line, we also assume that the first and the second normal derivative is prescribed at that triple line (as a constant along that line) and conclude directly that the unique interpolant from $\mathscr{P}(\Xi)$ matching the information at the triple point (picked out by our additional line) matches the given information on the entire triple line.

In this example, the orthogonal complement of a flat is only one-dimensional. The problem of choosing a minimal set of derivative information to be specified on a repeated flat becomes significantly more complicated when data are given on certain s-dimensional flats in $\mathbb{R}^d$ for $s < d-1$.

3. THE BASIC IDEA

Let $\mathbb{H}$ be a sequence of hyperplanes in $\mathbb{R}^d$, let $0 \leqslant s < d$, and recall from the Introduction the collection $\mathscr{M}_s(\mathbb{H})$ of all s-dimensional intersections of hyperplanes from $\mathbb{H}$. We are interested in polynomial interpolation to data given at all the flats in $\mathscr{M}_s(\mathbb{H})$.

For $s=0$, this is the interpolation problem addressed in [DR]. We propose to reduce the general case $s \geqslant 0$ to the known case $s=0$ by working with the larger sequence $\mathbb{H}_s$, obtained from $\mathbb{H}$ by adjoining to it s hyperplanes in general position with respect to it, and then considering interpolation at all the points in $\mathscr{M}_0(\mathbb{H}_s)$ to data there as derived from the data given on the flats in $\mathscr{M}_s(\mathbb{H})$. In this, we assume that the given data are *consistent*, i.e., provide unambiguous information at the points in $\mathscr{M}_0(\mathbb{H}_s)$. Since [DR] readily provides a suitable interpolant to data on $\mathscr{M}_0(\mathbb{H}_s)$, this reduces our task to showing that this interpolant does, indeed, match the data given on the flats in $\mathscr{M}_s(\mathbb{H})$.

If the hyperplanes in $\mathbb{H}$ are in general position, then $\mathscr{M}_0(\mathbb{H}_s)$ consists of exactly $\binom{\#\mathbb{H}_s}{d}$ points, i.e., the cardinality of $\mathscr{M}_0(\mathbb{H}_s)$ equals the dimension of the space

$$\Pi_k$$

of all polynomials of degree $\leqslant k$, with

$$k := \#\mathbb{H}_s - d = \#\mathbb{H} - (d-s).$$

More than that, in this case, Π_k is well known to contain a unique interpolant to arbitrary data given at the points in $\mathscr{M}_0(\mathbb{H}_s)$. In particular, let p be that interpolant from Π_k to data at $\mathscr{M}_0(\mathbb{H}_s)$ derived from the given data.

Then, assuming that the data are *compatible*, i.e., that the given datum p_M at a flat M in $\mathscr{M}_s(\mathbb{H})$ is the restriction to that flat of some element of Π_k, our interpolant p from Π_k is guaranteed to agree with p_M at sufficiently many points to force its restriction to M to coincide with p_M.

In the contrary case, various complications arise that will be dealt with fully later on. In the remainder of this introductory section, we now discuss just one such complication, namely the possibility that $\mathscr{M}_0(\mathbb{H}_s)$ consists of fewer than dim Π_k points but still each such point lies in exactly d of the hyperplanes from $\mathbb{H}_s$. The latter condition characterizes what we call in this paper the **simple case**. Already this case will provide the reader with a good feeling for the nature of the interpolation problem considered and some of the difficulties overcome by us in solving it.

When $\#\mathscr{M}_0(\mathbb{H}_s) < \dim \Pi_k$, we can only interpolate at $\mathscr{M}_0(\mathbb{H}_s)$ from some subspace of Π_k. Of the infinitely many choices possible, we take the subspace used in [DR], as this permits us to prove (later on) existence and uniqueness of an interpolant from that space to arbitrary (consistent and compatible) data given on $\mathscr{M}_s(\mathbb{H})$. For a description of that subspace, we find it more convenient to switch now, from the hyperplanes, to their normals and associated constants.

Precisely, we think, as we may, of $\mathbb{H}$ as having been obtained from a matrix X, with d rows and no null column, and a corresponding scalar sequence $(\lambda_x : x \in X)$ as the collection of hyperplanes

$$H_x := \{t \in \mathbb{R}^d : q_x(t) = 0\}$$

with

$$q_x\colon t \mapsto x \cdot t - \lambda_x,$$

and with x running over the columns of the matrix X. The relation $U \subset X$ we take to mean that U is obtained from X by deletion of some (or none) of its columns. Also, we denote by $\# X$ the length of the sequence X, i.e., the number of columns of the matrix X. The ordering of the columns of X is immaterial here. Because of the role such matrices play in box spline theory, we call them **direction sets** (in $\mathbb{R}^d$).

With this, we associate with each $U \subset X$ the following homogeneous polynomial of degree $\# U$,

$$\ell_U\colon t \mapsto \prod_{u \in U} (u \cdot t), \tag{3.1}$$

but write ℓ_x instead of $\ell_{\{x\}}$ or $\ell_{[x]}$ for $x \in X$. Further, we introduce the following subset of 2^X,

$$\mathbb{L}(X) := \{L \subset X : \operatorname{rank}(X \backslash L) = \operatorname{rank} X\}. \tag{3.2}$$

In these terms, [DR] show that (under the assumption that we are in the simple case with $s=0$) there is a unique interpolant to arbitrary values on the set $\mathcal{M}_0(\mathbb{H})$ from the polynomial space

$$\mathscr{P}(X) := \operatorname{span}\{\ell_L : L \in \mathbb{L}(X)\}. \tag{3.3}$$

Note that $\mathscr{P}(X) = \Pi_{\# X-d}$ in case $\mathbb{H}$ is in general position (provided that there are at least d hyperplanes in the sequence). For more information on $\mathscr{P}(X)$, see Section 4 below.

Now let Y be a sequence of s directions, or, equivalently, a $(d \times s)$-matrix of directions associated with the s hyperplanes in general position adjoined to $\mathbb{H}$ to obtain $\mathbb{H}_s$, and continue to assume that we are in the simple case, i.e., each point in $\mathcal{M}_0(\mathbb{H}_s)$ lies in exactly d of the hyperplanes in $\mathbb{H}_s$. Set

$$Z := X \cup Y.$$

Then [DR] provides a unique interpolant from $\mathscr{P}(Z)$ to arbitrary values at the points of $\mathcal{M}_0(\mathbb{H}_s)$. We will show below that

$$\mathscr{P}(Z) = \mathscr{P}_s(X) := \mathscr{P}(X)\, \Pi_r := \operatorname{span}\{pq : p \in \mathscr{P}(X), q \in \Pi_r\},$$

with $r := (s-d+\operatorname{rank}(X))_+$. In particular, $\mathscr{P}(Z)$ only depends on X and s.

Assume now that we have been given **consistent** data on the flats in $\mathcal{M}_s(\mathbb{H})$, i.e., data

$$(p_M : M \in \mathcal{M}_s(\mathbb{H}))$$

so that, for every $M_1, M_2 \in \mathcal{M}_s(\mathbb{H})$, the polynomials p_{M_1} and p_{M_2} coincide on $M_1 \cap M_2 \cap \mathcal{M}_0(\mathbb{H}_s)$. Such consistency is clearly necessary if we are to construct an interpolant to these data.

For each $\theta \in \mathcal{M}_0(\mathbb{H}_s)$, there exists some basis $B \subset Z$ such that $\theta \in H_x$ for every $x \in B$. Since $\# Y = s$, $X \cap B$ contains some B' of length $d-s$. Therefore, $M := \bigcap_{x \in B'} H_x$ is a flat in $\mathcal{M}_s(X)$ that contains θ, hence the derived datum

$$a_\theta := p_M(\theta)$$

is well-defined. Also, by the assumed consistency, this definition is independent of our choice of M.

It follows from [DR, Theorem 7.1] (see Theorem 5.11 below) that there is exactly one element of $\mathscr{P}(Z)$ that matches these data $(a_\theta : \theta \in \mathcal{M}_0(\mathbb{H}_s))$. To show that, for all $M \in \mathcal{M}_s(\mathbb{H})$, this element also matches p_M on M takes additional work; see the proof of Theorem 5.10 below. In particular, for such a conclusion, we need to assume that the given data are ***X*-compatible** in the sense that each p_M is the restriction to M of some element of $\mathscr{P}(Z)$.

But the above discussion already makes clear that an interpolant from $\mathscr{P}(Z)$ to any X-compatible and consistent data on $\mathscr{M}_s(\mathbb{H})$ is unique.

4. THE SPACE $\mathscr{P}_s(X)$

In this preparatory section, we discuss the polynomial space $\mathscr{P}_s(X)$, to be used eventually as the space of interpolants to information given on certain flats determined by X.

We start off with a **direction set**, as introduced in the preceding section, i.e., a matrix X with d rows and no null columns. We denote the column span of X by

$$\text{ran } X.$$

Recall from (3.3) the polynomial space

$$\mathscr{P}(X) := \text{span}\{\ell_L : L \in \mathbb{L}(X)\} \tag{4.1}$$

with

$$\ell_U : t \mapsto \prod_{x \in U} (x \cdot t), \tag{4.2}$$

and

$$\mathbb{L}(X) := \{L \subset X : \text{rank}(X \backslash L) = \text{rank } X\}. \tag{4.3}$$

This polynomial space naturally arises in box spline problems. It reflects in its structure much of the geometry of the multiset X, and was independently discovered by several authors ([HS1, J, DR]; some of us regret that this is the chronological order; see also [DM]).

Now note that ℓ_L is a homogeneous polynomial of (exact) degree $\#L$, and is constant in all directions perpendicular to ran L, i.e., $D_z \ell_L = 0$ for all $z \perp \text{ran } L$. Since each $L \in \mathbb{L}(X)$ can have at most $\#X - \text{rank } X$ elements, it is clear that $\mathscr{P}(X)$ is a dilation-invariant subspace of $\Pi_{\#X - \text{rank } X}(\text{ran } X)$, with

$$\Pi_k(M) \subseteq \Pi$$

the subspace of all polynomials of degree $\leqslant k$ on $\mathbb{R}^d$ that are constant in all directions orthogonal to the flat M. It is also clear that $\mathscr{P}(X) = \mathscr{P}(X \cup B)$, with B any basis for a linear subspace complementary to ran X since $\mathbb{L}(X) = \mathbb{L}(X \cup B)$ for any such B. The dimension of $\mathscr{P}(X)$ is known to equal the number of submatrices of X that are bases for ran X. For more information, see, e.g., [DR]. It is important to note that if X is in **general position** (i.e., if every $U \subset X$ with $\# U \leqslant \operatorname{rank} X$ is 1-1), then the space $\mathscr{P}(X)$ *coincides* with $\Pi_{\# X - \operatorname{rank} X}(\operatorname{ran} X)$.

We are ready for the definition of $\mathscr{P}_s(X)$.

DEFINITION 4.4. Let X be a direction set and let s be an integer. Then

$$\mathscr{P}_s(X) := \mathscr{P}(X)\, \Pi_{(s-d+\operatorname{rank}(X))_+}. \tag{4.5}$$

Note that $\mathscr{P}_s(X) = \mathscr{P}(X)$ for $s \leqslant d - \operatorname{rank}(X)$, and that the sequence $(\mathscr{P}_s(X) : s = 0, 1, 2, ...)$ is nested, i.e., $\mathscr{P}_{s'}(X) \subseteq \mathscr{P}_s(X)$ whenever $s' < s$.

In the sequel, the following characterization of $\mathscr{P}_s(X)$ will be important since it shows the characterization of $\mathscr{P}(\Xi)$ from [DR] to be applicable here, and shows that, for Y in general position, $\mathscr{P}(X \cup Y)$ only depends on X and $\# Y$:

PROPOSITION 4.6. *Let X and Y be direction sets in $\mathbb{R}^d$, with $X \cup Y$ of full rank, and $s := \# Y$. Then*

$$\mathscr{P}(X \cup Y) \subseteq \mathscr{P}_s(X),$$

with equality if (and only if) Y is **in general position with respect to** *X, i.e., no $y \in Y$ lies in a proper subspace spanned by elements of $(X \cup Y) \backslash y$.*

Proof. Since $X \cup Y$ is of full rank by assumption, Y must contain some basis B for a subspace complementary to $\operatorname{ran} X$. Then $X \cup B$ is of full rank, $\mathscr{P}(X \cup B) = \mathscr{P}(X)$ and $s - d + \operatorname{rank}(X) = \#(Y \backslash B)$. We may therefore assume without loss that already X is of full rank, hence

$$\mathscr{P}_s(X) = \mathscr{P}(X)\, \Pi_s.$$

Given $L \in \mathbb{L}(X \cup Y)$, we let $k := \#(L \cap Y)$. Then the rank-d matrix $(X \cup Y) \backslash L$ contains exactly $s - k$ elements from Y, and thus $\operatorname{rank}(X \backslash L) \geqslant d - (s - k)$. Since $\operatorname{rank} X = d$, there exists $Z \subset L \cap X$ with $\# Z \leqslant s - k$ such that $\operatorname{rank}((X \backslash L) \cup Z) = d$. Now,

$$\ell_L = \ell_{L \backslash (Y \cup Z)} \ell_{(L \cap Y) \cup Z}, \tag{4.7}$$

and we have that $\#((L \cap Y) \cup Z) \leqslant k + (s - k) = s$, hence $\ell_{(L \cap Y) \cup Z} \in \Pi_s$. At the same time, $L \backslash (Y \cup Z)$ is a subset of X, and its complement in X is $(X \backslash L) \cup Z$ which is known to be of full rank. This means that $L \backslash (Y \cup Z) \in \mathbb{L}(X)$, and therefore $\ell_{L \backslash (Y \cup Z)} \in \mathscr{P}(X)$. Consequently, we infer from (4.7) that $\ell_L \in \mathscr{P}(X)\, \Pi_s$. This being true for every $L \in \mathbb{L}(X \cup Y)$, we conclude that a spanning set for $\mathscr{P}(X \cup Y)$ lies in $\mathscr{P}_s(X)$, and hence $\mathscr{P}(X \cup Y) \subset \mathscr{P}_s(X)$.

For the converse inclusion, let $L \in \mathbb{L}(X)$, and let $B \subset (X \backslash L)$ be a basis for $\mathbb{R}^d$ (i.e., a square invertible $d \times d$ matrix) The existence of such a B is guaranteed by the definition of $\mathbb{L}(X)$ and our assumption that rank $X = d$. Let $Z := B \cup Y$. Then $\#Z = s + d$, and, by our assumption on Y, Z is in general position, hence $\mathscr{P}(Z) = \Pi_s$. This implies that, given an element of the form $\ell_L q$ with $q \in \Pi_s$, we are able to write this polynomial in the form

$$\ell_L q = \sum_{L' \in \mathbb{L}(B \cup Y)} c_{L'} \ell_{L \cup L'}$$

for certain scalars $c_{L'}$. Each $L \cup L'$ in this sum lies in $\mathbb{L}(X \cup Y)$ since $(X \cup Y) \backslash (L \cup L') \supset (B \cup Y) \backslash L'$, and the latter is of rank d because $L' \in \mathbb{L}(B \cup Y)$. Consequently, $\ell_L q \in \mathscr{P}(X \cup Y)$, and we have thus proved that a spanning set for $\mathscr{P}_s(X)$ lies in $\mathscr{P}(X \cup Y)$, and hence $\mathscr{P}_s(X) \subset \mathscr{P}(X \cup Y)$.

Finally, the "only if" assertion follows from the above proof and the known formula for the dimension of $\mathscr{P}(X)$ as follows. Suppose that Y is *not* in general position with respect to X, and let Z be a set of s directions that *is* in general position with respect to X. Then, $X \cup Z$ contains more bases than does $X \cup Y$. Since this basis count determines the dimension of the corresponding $\mathscr{P}$-space, it follows that $\dim \mathscr{P}(X \cup Y) < \dim \mathscr{P}(X \cup Z)$. However, we proved above that $\mathscr{P}(X \cup Z) = \mathscr{P}_s(X)$, whence the desired conclusion. ∎

5. THE INTERPOLATION PROBLEM: THE SIMPLE CASE

Recall that we introduced the direction set X as a sequence or matrix of normal vectors, one for each of the hyperplanes

$$H_x := \{t \in \mathbb{R}^d : x \cdot t = \lambda_x\}, \qquad x \in X,$$

in the given sequence $\mathbb{H}$, with suitable constants $\lambda_x \in \mathbb{R}$. These constants will be held fixed throughout the discussion. With $s \in \{d - \text{rank}(X), ..., d-1\}$ fixed, we wish to interpolate to polynomial information given on the collection

$$\mathscr{M}_s(X) := \left\{ \bigcap_{x \in U} H_x : U \subset X, \text{ rank } U = d - s \right\} \tag{5.1}$$

of all flats of dimension s that can be expressed as the intersection of hyperplanes from $(H_x : x \in X)$. Note that we have now switched, from the notation $\mathscr{M}_s(\mathbb{H})$ to the (less precise) notation $\mathscr{M}_s(X)$. Since the hyperplanes $(H_x : x \in X)$ depend on $(\lambda_x : x \in X)$, so does the set $\mathscr{M}_s(X)$, but we have suppressed the λ subscript here, as we did with H_x, since these constants are being held fixed throughout. However, we will write

$$\mathscr{M}_{s,0}(X)$$

instead of $\mathscr{M}_s(X)$ when we want to stress the fact that all hyperplanes contain the origin, i.e., in the determination of these s-dimensional flats, all λ_x were chosen to be zero.

For each $M \in \mathscr{M}_s(X)$, we consider the following subset X^M of X,

$$X^M := (x \in X : M \subset H_x). \tag{5.2}$$

By the definition of $\mathscr{M}_s(X)$, rank $X^M = d - s$, and thus $\# X^M \geqslant d - s$.

Definition 5.3. Let X, $\lambda = (\lambda_x : x \in X)$, and $\mathscr{M}_s(X)$ be as above. We call the pair (X, λ) **simple** if each $M \in \mathscr{M}_{d - \text{rank } X}(X)$ is contained in no more than rank X hyperplanes from $\mathbb{H}$, i.e., if $\# X^M = \text{rank } X$ for each such M. In that case, we also call the corresponding interpolation problem **simple**.

For instance, in the example in Section 2, the Cases 1.1–1.3 are simple, while the Cases 2.1–2.4 are not. Note that $\mathscr{M}_{d - \text{ran } X}(X)$ is just the pointset $\mathscr{M}_0(X)$ in case X is of full rank and has at least d columns. In other words, if rank $X = d$, then simplicity means that each $\theta \in \mathscr{M}_0(X)$ is the intersection of exactly d hyperplanes in $\mathbb{H}$.

For the rest of this section, we assume that our interpolation problem is simple. Note that this assumption is entirely on the constants $(\lambda_x : x \in X)$ and not on the matrix X.

We now assume that, with each flat $M \in \mathscr{M}_s(X)$, we are given a polynomial p_M on M, i.e., the restriction $p|_M$ of some $p \in \Pi$, and it is this polynomial information we hope to match by some element of $\mathscr{P}_s(X)$. We will not be able to accomplish this unless

$$p_M \in \mathscr{P}_s(X)|_M. \tag{5.4}$$

Surprisingly, this simple necessary condition for the existence of a solution to our problem (along with the obvious consistency condition discussed below) is also sufficient, leading to the following.

Definition 5.5. The data $(p_M : M \in \mathscr{M}_s(X))$ are termed X-**compatible** if (5.4) holds for every $M \in \mathscr{M}_s(X)$.

We now take time out to study this notion of compatibility in some depth. The analysis of our interpolation problem is resumed after Proposition 5.8.

In order to characterize X-compatible data and for later use, we next prove that, on M, $\mathscr{P}_s(X)$ coincides with $\mathscr{P}(Z_M)$ for a certain direction set Z_M. The construction of Z_M involves the orthogonal projector

$$P_M$$

onto the linear subspace $M-M$ parallel to the flat M. Explicitly,

$$Z_M := P_M(Z \backslash Z^M) \cup B_M,$$

with

$$Z := X \cup Y, \tag{5.6}$$

with Y an arbitrary s-set of directions in general position with respect to X, and with B_M any basis from Z^M for the orthogonal complement

$$M^{\perp} := \{t \in \mathbb{R}^d : t \perp (M-M)\}$$

of M in $\mathbb{R}^d$. Having made appropriate choices for Y and B_M, we now keep them fixed for the remainder. Note that, necessarily, $\mathscr{P}(Z_M) \subset \Pi(M)$. The exclusion of Z^M is needed since P_M maps all of Z^M to 0. Here is the relevant formal statement and its proof.

LEMMA 5.7. *Let Z be a direction set, and let $M \in \mathscr{M}_s(Z)$. Then*

$$\mathscr{P}(Z)|_M = \mathscr{P}(Z_M)|_M.$$

In particular, with Y an s-direction set in general position with respect to X,

$$\mathscr{P}_s(X)|_M = \mathscr{P}((X \cup Y)_M)|_M, \qquad \forall M \in \mathscr{M}_s(X).$$

Proof. The map

$$L \mapsto \tilde{L} := P_M(L \backslash Z^M)$$

carries $\mathbb{L}(Z)$ *onto* $\mathbb{L}(Z_M)$. Indeed, P_M carries a spanning set of $\mathbb{R}^d$ to a spanning set of $M-M$, hence $P_M(Z) \backslash \tilde{L} = P_M(Z \backslash L)$ spans $M-M$. Then $P_M(Z \backslash Z^M) \backslash \tilde{L}$ spans $M-M$, too, since it differs from $P_M(Z \backslash L)$ by $P_M(Z^M) = \{0\}$. Since $Z_M \backslash \tilde{L} = P_M(Z \backslash L) \cup B_M$, and since B_M spans $M^{\perp}$, we conclude that $\tilde{L} \in \mathbb{L}(Z_M)$.

To show that the map is *onto*, let $K \in \mathbb{L}(Z_M)$. Since Z_M contains exactly $d-s$ elements (viz., the elements of B_M) not contained in the s-dimensional $M-M$, and since $Z_M \backslash K$ spans $\mathbb{R}^d$, K must be disjoint from B_M. Hence K lies in $P_M(Z \backslash Z^M)$. In particular, $K = P_M(L)$ for some $L \subset Z \backslash Z^M$. On the other hand, $Z_M \backslash K$ spans $\mathbb{R}^d$, and is the union of B_M and $P_M(Z \backslash Z^M) \backslash K$. Therefore, since $\operatorname{rank}(B_M) = d-s$, we have that $\operatorname{rank}(P_M(Z \backslash Z^M) \backslash K) \geqslant s$, *a fortiori* $\operatorname{rank}(Z\backslash(Z^M \cup L)) \geqslant s$. It easily follows then that $\operatorname{rank}(Z\backslash L) = d$, hence that $L \in \mathbb{L}(Z)$.

With this, let $L \in \mathbb{L}(Z)$ and consider ℓ_L on M. If $t \in M$, then

$$\ell_L(t) = \prod_{x \in L} (x \cdot t) = \prod_{x \in L} ((P_M x) \cdot t + (x - P_M x) \cdot t),$$

and $c_x := (x - P_M x) \cdot t$ is a *constant* on M. This shows that, on M, ℓ_L agrees with some scalar multiple of the function $f\colon t \mapsto \prod_{\tilde{x} \in \tilde{L}} (\tilde{x} \cdot t + c_x)$, and, since $\tilde{L} \in \mathbb{L}(Z_M)$, such f is in $\mathscr{P}(Z_M)$ (as a linear combination of polynomials of the form ℓ_K, $K \in \mathbb{L}(X)$). Consequently, $\mathscr{P}(Z)|_M \subseteq \mathscr{P}(Z_M)|_M$. The converse containment is obtained in an analogous way.

The second equality in the lemma is a consequence of the first and of Proposition 4.6. ∎

We recall from [BDR, (2.22)] that, for a full-rank direction set Ξ, $p \in \Pi$ is in $\mathscr{P}(\Xi)$ iff, for every $1 \leqslant r \leqslant d$ and every $N \in \mathscr{M}_{r,0}(\Xi)$ and every $t \in \mathbb{R}^d$,

$$\deg (p|_{t+N}) \leqslant \#(\Xi \backslash N^\perp) - r.$$

After identifying the datum p_M given on $M \in \mathscr{M}_s(X)$ with its unique extension to an element of $\Pi(M) \subset \Pi$, we therefore obtain, with the aid of Lemma 5.7, the following.

Proposition 5.8. *The data $(p_M : M \in \mathscr{M}_s(X))$ are X-compatible if and only if, for $r \leqslant s$ and every $N \in \mathscr{M}_{r,0}(Z_M)$ and every $t \in M - M$,*

$$\deg (p_M|_{t+N}) \leqslant \#(Z_M \backslash N^\perp) - r = \#(X_M \backslash N^\perp) + s - r.$$

We now resume the discussion of our interpolation problem. We want to interpolate all the data $(p_M : M \in \mathscr{M}_s(X))$ by some $p \in \mathscr{P}_s(X)$, hence must ensure also that these data are consistent enough to guarantee the existence of a smooth interpolant at least locally. These conditions, which we refer to as "the consistency conditions", are very natural and simple in the present case.

Definition 5.9. We say that the data $(p_M : M \in \mathscr{M}_s(X))$ are **consistent** if, for every $M_1, M_2 \in \mathscr{M}_s(X)$, the polynomials p_{M_1} and p_{M_2} coincide on $M_1 \cap M_2 \cap \mathscr{M}_0(Z)$ (with Z as in (5.6)).

Theorem 5.10. *Assume that (X, λ) is simple, and let $(p_M : M \in \mathscr{M}_s(X))$ be consistent and X-compatible data. Then there exists exactly one $p \in \mathscr{P}_s(X)$ that interpolates these data, i.e., that satisfies*

$$p|_M = p_M, \qquad \forall M \in \mathscr{M}_s(X).$$

If $s = 0$, then the information is given at points. In this case, $\mathscr{P}_s(X) = \mathscr{P}(X)$, and both the consistency and compatibility conditions are vacuous. Thus, for the case $s = 0$, our theorem reads as follows:

THEOREM 5.11 [DR]. *Any data given on the pointset $\mathscr{M}_0(X)$ is interpolated by a unique element in $\mathscr{P}(X)$.*

The existence part of this theorem can be easily proved by finding in $\mathscr{P}(X)$ Lagrange polynomials for the data, i.e., polynomials that vanish at all points in $\mathscr{M}_0(X)$ but one. The uniqueness is harder and can be proved by showing that $\dim \mathscr{P}(X) = \# \mathscr{M}_0(X)$. We omit all these details since Theorem 5.11 has already been proved in [DR]; it is a special case of Theorem 7.1 there, which covers also the general case for $s = 0$, Theorem 7.4, here. The construction of the Lagrange polynomials together with the fact that these polynomials form a basis for $\mathscr{P}(X)$ is the content of Theorem 4.1 of [DR].

We now reduce the general Theorem 5.10 to the known Theorem 5.11:

Proof of Theorem 5.10. Recall the direction set $Y \in \mathbb{R}^{d \times s}$ in general position with respect to X chosen earlier and the notation $Z := X \cup Y$, and the fact that, by Proposition 4.6, $\mathscr{P}_s(X) = \mathscr{P}(Z)$. We associate with each $y \in Y$ a constant λ_y such that also $\mathscr{M}_0(Z)$ is simple, i.e., such that exactly d hyperplanes H_z, $z \in Z$, contain a given point $\theta \in \mathscr{M}_0(Z)$. Our proof then proceeds in two steps: the first one, we already took in Section 3, where we used the assumed consistency to show that the given information $(p_M : M \in \mathscr{M}_s(X))$ determines *uniquely* data values $(a_\theta : \theta \in \mathscr{M}_0(Z))$, and then invoked Theorem 5.11 to find exactly one $p \in \mathscr{P}(Z) = \mathscr{P}_s(X)$ that interpolates these data, thus concluding uniqueness of the interpolant. In the second step, we show that, for every $M \in \mathscr{M}_s(X)$, the fact that the interpolant p coincides with p_M on $M \cap \mathscr{M}_0(Z)$ implies that $p = p_M$ on all of M, thus showing existence of the interpolant.

Here are the details for that second step.

To prove the existence, we fix $M \in \mathscr{M}_s(X)$ and wish to show that the interpolant p coincides on M with p_M. Now, p_M and $p|_M$ agree on $\mathscr{M}_0(Z) \cap M$, and both are in $\mathscr{P}(Z)|_M$ (the former by the assumed X-compatibility and the latter by construction), while $\mathscr{P}(Z)|_M = \mathscr{P}(Z_M)|_M$, by Lemma 5.7. Further, by Theorem 5.11 (easily applied to the current situation by an affine change of variables), the space $\mathscr{P}(Z_M)$ contains a unique interpolant to arbitrary data at the point set $\widetilde{\mathscr{M}}(Z_M)$, with the tilde indicating that the zero-dimensional flats are constructed from hyperplanes $H_{\tilde{x}} = \{t : q_{\tilde{x}}(t) = 0\}$, $\tilde{x} \in Z_M$, and the corresponding constant $\lambda_{\tilde{x}}$ in the linear polynomial $q_{\tilde{x}}: t \mapsto \tilde{x} \cdot t - \lambda_{\tilde{x}}$ chosen in such a way that $q_{\tilde{x}}$ agrees on M with the polynomial q_x, as is done in the proof of Lemma 5.7. Thus, we can conclude that $p|_M = p_M$ (and so declare Theorem 5.10 proved), once we prove that

$$\mathscr{M}_0(Z) \cap M = \widetilde{\mathscr{M}}_0(Z_M). \tag{5.12}$$

For this, $\theta \in \widetilde{\mathscr{M}}_0(Z_M)$ iff the submatrix Z_M^θ of all directions $\tilde{x} \in Z_M$ with $\theta \in H_{\tilde{x}}$ contains a basis for $\mathbb{R}^d$. Any basis in Z_M necessarily contains the basis B_M for $M^\perp$, hence $\widetilde{\mathscr{M}}_0(Z_M) \subset M$. For any other element $\tilde{x}$ of such a basis, we have $H_{\tilde{x}} \cap M = H_x \cap M$, by the construction of $H_{\tilde{x}}$ just detailed, hence θ also lies in the corresponding H_x. Consequently, $\mathscr{M}_0(Z) \cap M \supset \widetilde{\mathscr{M}}_0(Z_M)$.

Conversely, $\theta \in \mathscr{M}_0(Z) \cap M$ iff the submatrix Z^θ of all directions $x \in Z$ with $\theta \in H_x$ contains some basis B for $\mathbb{R}^d$ and, within that basis, a basis for $M^\perp$ taken from X^M. For any element $x \in B \backslash X^M$, $q_x = q_{\tilde{x}}$ on M, hence $\theta \in \widetilde{\mathscr{M}}_0(Z_M)$. This finishes the proof of (5.12) and, thereby, the proof of the theorem. ∎

6. SOME FACTS ABOUT POLYNOMIALS

Some of the theorems (in the next section) that are needed for proving our main result make claims of the form

$$F \subset G, \tag{6.1}$$

with both F and G a **polynomial space** (i.e., a linear subspace of Π), but not necessarily of finite dimension nor of finite codimension. However, in all cases of interest to us, the spaces F and G are **D-invariant** i.e., invariant under differentiation in any direction, hence invariant under any constant-coefficient differential operator $p(D) := \sum_\alpha (D^\alpha p(0)/\alpha!)\, D^\alpha$, taken as a map on Π, with $D^\alpha := \prod_{i=1}^d D_i^{\alpha(i)}$, and D_i differentiation with respect to the ith argument.

For D-invariant F and G, one may try to prove the inclusion $F \subset G$ by inspecting the constant-coefficient differential operators that annihilate these spaces, since, for an arbitrary subset G of Π, the set

$$\mathscr{I}_G := \{ p \in \Pi : G \subset \ker p(D) \}$$

is an ideal (hence has a *finite* generating set) and, further,

$$G \subset \ker \mathscr{I}_G := \bigcap_{p \in \mathscr{I}_G} \ker p(D) = \bigcap_{p \in G_0} \ker p(D)$$

for any generating set G_0 for $\mathscr{I}_G$, with equality if and only if G is a **co-ideal**, i.e., a D-invariant polynomial space that is closed in the weak topology induced by the pairing

$$\Pi \times \Pi \colon (p, q) \mapsto \langle p, q \rangle := \sum_\alpha D^\alpha p(0)\, D^\alpha q(0)/\alpha!.$$

Hence, if G is a co-ideal, then we can conclude (6.1) as soon as we know that, for all p in some generating set for the ideal $\mathscr{I}_G$, $p(D)$ annihilates F. For, then we know that $\mathscr{I}_G \subset \mathscr{I}_F$, hence

$$F \subset \ker \mathscr{I}_F \subset \ker \mathscr{I}_G = G.$$

In our particular applications, G will be a **homogeneous** polynomial space, i.e., is spanned by homogeneous polynomials (or, equivalently, is invariant under dilations). For that case, we have the following simple, yet very useful, observation:

PROPOSITION 6.2. *Any sum of homogeneous D-invariant polynomial spaces is a co-ideal.*

Proof. Since the sum of homogeneous D-invariant polynomial spaces is also homogeneous and D-invariant, we need only to prove the case when there is a single summand, F, in the sum.

Since we assume that F is D-invariant, we need only to prove that it is closed. Let (f_n) be a sequence in F weakly convergent to $f \in \Pi$. Then $f \in \Pi_k$ for some k and so, necessarily, already

$$f_n^{[k]} =: \sum_{|\alpha| \leqslant k} \frac{D^\alpha f_n(0)}{\alpha!} (\ )^\alpha$$

converges weakly to f, while, by the homogeneity of F, each $f_n^{[k]}$ is in the finite-dimensional space $F \cap \Pi_k$, hence so must f be. ∎

We conclude this section with two further simple observations of use in the next section.

PROPOSITION 6.3. *Let F and G be co-ideals, with $\mathscr{I}_G = \mathscr{I}(\Gamma)$, and let $p \in \Pi$. Then, $p \in F + G$ if and only if there exists $f \in F$ so that, for all $\gamma \in \Gamma$, $\gamma(D)(p - f) = 0$.*

For our last statement, let M be a flat in $\mathbb{R}^d$ and recall our notation P_M for the orthogonal projector onto $M - M$. Then,

$$D_y p = D_{P_M y}\, p, \qquad \forall y \in \mathbb{R}^d, \quad \forall p \in \Pi(M). \tag{6.4}$$

Indeed, since $p \in \Pi(M)$ is constant in all directions perpendicular to M, we have $D_{y - P_M y} p = 0$.

In particular, D_y maps $\Pi(M)$ *onto* itself unless $P_M y = 0$, i.e., unless $y \in M^\perp$. Since $D_y(qp) = (D_y q)\, p + q(D_y p)$, this implies (by induction on $\deg q$) the following

PROPOSITION 6.5. *For any flat M in $\mathbb{R}^d$, any $y \in \mathbb{R}^d \backslash M^\perp$, and any D-invariant linear subspace F of Π,*

$$D_y(F\Pi(M)) = F\Pi(M).$$

7. THE INTERPOLATION PROBLEM: THE GENERAL CASE

The basic set-up in the general case is the same as in the simple case: we are given the direction set X and the constants $(\lambda_x : x \in X)$ and want to interpolate from $\mathscr{P}_s(X)$ to polynomial information given on the flats in $\mathscr{M}_s(X)$. Only that at this time, we no longer assume simplicity, i.e., while still for every $M \in \mathscr{M}_s(X)$ the set X^M (of all vectors whose corresponding hyperplane contains M) spans $M^\perp$, there might be more than $d - s$ vectors in X^M, which is to say that the flat M appears with some multiplicity. Therefore, the initial task is to define precisely the multiplicity notion. This will eventually determine the type of interpolation conditions we expect to satisfy.

Before embarking on the precise definition of "multiplicity" here, we make the following simple count: suppose that $M \in \mathscr{M}_s(X)$ is given and assume that X^M contains more than $d - s$ vectors. In this case, the interpolating polynomial should match not only function values given on M but also some prescribed derivatives on M, i.e., we expect to be given data

$$(p_{M,\varphi} : \varphi \in \Phi)$$

consisting of polynomials on M, with Φ some M-dependent polynomial space that describes the derivatives to be interpolated, in the sense that we require our interpolant p to satisfy all of the conditions

$$(\varphi(D)\, p)|_M = p_{M,\varphi}, \qquad \forall \varphi \in \Phi.$$

Of course, some consistency requirements must be enforced; for example, it should be assumed that

$$\alpha p_{M,\varphi} + \beta p_{M,\psi} = p_{M,\alpha\varphi + \beta\psi}, \qquad \forall \varphi, \psi \in \Phi, \quad \forall \alpha, \beta \in \mathbb{C}.$$

As we will see in a moment, the exact definition of the space Φ is fairly complicated, but one thing can be observed easily in advance: *the dimension of Φ should be the number of bases for $M^\perp$ that can be extracted from X^M* (i.e., the number of submatrices of X^M of length *and* rank $d-s$). This is so because this number counts the number of flats in $\mathscr{M}_s(X)$ that have been merged into the one M while passing from the generic or simple case to the present general case.

We now define exactly the type of derivatives that should be interpolated. For this purpose, we recall the polynomial space $\mathscr{D}(\Xi)$ which is also intimately related to box spline theory:

DEFINITION 7.1. Let Ξ be any matrix with d rows. Let $\mathbb{K}(\Xi)$ be the complement of $\mathbb{L}(\Xi)$ in 2^Ξ, i.e.,

$$\mathbb{K}(\Xi) := \{K \subset \Xi : \operatorname{rank}(\Xi \backslash K) < \operatorname{rank} \Xi\}. \tag{7.2}$$

Then the polynomial space $\mathscr{D}(\Xi) \subset \Pi(\operatorname{ran} \Xi)$ is defined as the joint kernel of the differential operators on $\Pi(\operatorname{ran} \Xi)$ induced by $\mathbb{K}(\Xi)$:

$$\begin{aligned} \mathscr{D}(\Xi) &:= \{\varphi \in \Pi(\operatorname{ran} \Xi) : \ell_K(D)\, \varphi = 0, \forall K \in \mathbb{K}(\Xi)\} \\ &= \ker \mathscr{I}(\ell_K : K \in \mathbb{K}(\Xi)) \subset \Pi(\operatorname{ran} \Xi). \end{aligned}$$

In particular,

$$\mathscr{D}(\Xi) = \mathscr{D}(\Xi \cup B)$$

for any basis B for $(\operatorname{ran} \Xi)^\perp$.

We are now ready to describe the information to be interpolated: we assume that the data consist of polynomials

$$p_{M,\varphi} \in \Pi(M), \qquad M \in \mathscr{M}_s(X), \quad \varphi \in \mathscr{D}(X^M),$$

and that the interpolant p should then satisfy

$$(\varphi(D)\, p)|_M = p_{M,\varphi}, \qquad \forall M \in \mathscr{M}_s(X), \quad \forall \varphi \in \mathscr{D}(X^M).$$

It may be helpful for the reader to consider briefly the spaces $\mathscr{D}(X^M)$ that occur in the example in Section 2. In Cases 1.1–1.3 and 2.1, each X^M has just one column, $X^M = [x]$ say, and, correspondingly, $\mathbb{K}(X^M) = \{[x]\}$, hence $\mathscr{D}(X^M) = \Pi_0$. In Cases 2.2 and 2.3, one of the M has $X^M = [x, x]$ for a certain x, hence now $\mathbb{K}(X^M) = \{[x, x]\}$, and therefore $\mathscr{D}(X^M) = \Pi_1(\operatorname{ran}[x]) = \operatorname{span}\{1, \ell_x\}$. Finally, Cases 2.4 is quite similar in that the one and only M has $X^M = X = [x, x, x]$, hence $\mathscr{D}(X^M) = \operatorname{span}\{1, \ell_x, \ell_x^2\}$ in this case. As we said at the end of Section 2, one needs to go to higher dimensions in order to fully appreciate the power of the construct $\mathscr{D}(X^M)$.

We postpone the discussion concerning the *consistency* of the data, but we can already present our compatibility requirements for the data with $\mathscr{P}_s(X)$:

DEFINITION 7.3. We say that the given data $(p_{M,\varphi} : M \in \mathscr{M}_s(X), \varphi \in \mathscr{D}(X^M))$ are *X-compatible* if

$$p_{M,\varphi} \in (\mathscr{P}_s(X))|_M, \qquad \forall M \in \mathscr{M}_s(X), \quad \forall \varphi \in \mathscr{D}(X^M).$$

While one might have expected here the stronger condition $p_{M,\varphi} \in (\varphi(D)\, \mathscr{P}_s(X))|_M$, the given condition turns out to suffice.

So, with the notion of consistency yet to be defined, we assume that we are given consistent information $(p_{M,\varphi} : M \in \mathscr{M}_s(X), \varphi \in \mathscr{D}(X^M))$ that is X-compatible and want to prove the existence and uniqueness of $p \in \mathscr{P}_s(X)$ that matches these data. As in the simple case, our solution method is based on the reduction of this problem to the case $s = 0$ and uses the known results for this case that were established in Theorem 7.1 of [DR], as follows.

THEOREM 7.4 [DR]. *Let Ξ be a direction set in $\mathbb{R}^d$, of rank d, and let $(H_x : x \in \Xi)$ be a corresponding sequence of hyperplanes, each perpendicular to its associated x. For each $\theta \in \mathbb{R}^d$, let Ξ^θ be defined as*

$$\Xi^\theta := (x \in \Xi : \theta \in H_x),$$

and let $\mathscr{M}_0(\Xi)$ be the set of all θ with rank $\Xi^\theta = d$. *Then, for every smooth function f: $\mathbb{R}^d \to \mathbb{C}$, there exists exactly one $p \in \mathscr{P}(\Xi)$ that satisfies*

$$\varphi(D)\, p(\theta) = \varphi(D)\, f(\theta), \qquad \forall \theta \in \mathscr{M}_0(\Xi), \quad \forall \varphi \in \mathscr{D}(\Xi^\theta).$$

The smooth function f in this theorem serves only to ensure the consistency of the data $(\varphi(D)\, f(\theta) : \theta \in \mathscr{M}_0(\Xi), \varphi \in \mathscr{D}(\Xi^\theta))$. We could have replaced each value $\varphi(D)\, f(\theta)$ by a number $p_{\theta,\varphi}$ and required the consistency conditions

$$\alpha p_{\theta,\varphi} + \beta p_{\theta,\psi} = p_{\theta,\alpha\varphi+\beta\psi}, \qquad \theta \in \mathscr{M}_0(\Xi), \quad \varphi, \psi \in \mathscr{D}(\Xi^\theta), \quad \alpha, \beta \in \mathbb{C},$$

since these conditions are equivalent to the existence of a smooth interpolant to the data.

With the aid of Theorem 7.4, we treat the case $s > 0$ as follows: We assume that $\mathscr{M}_s(X)$ is not empty (since otherwise there is nothing to prove), i.e., we assume that rank $X \geqslant d - s$, and add to X, as in the simple case, s vectors Y that are in general position relative to X, thus obtaining the direction set

$$Z := X \cup Y,$$

of full rank. Since, by Proposition 4.6, $\mathscr{P}_s(X) = \mathscr{P}(Z)$, we seek our interpolant from $\mathscr{P}(Z)$. Precisely, we will derive from the data $(p_{M,\varphi} : M \in \mathscr{M}_s(X),\ \varphi \in \mathscr{D}(X^M))$ uniquely determined data $(p_{\theta,\varphi} : \theta \in \mathscr{M}_0(Z),\ \varphi \in \mathscr{D}(Z^\theta))$ and show that the unique interpolant in $\mathscr{P}(Z)$ to the latter data (which is provided by Theorem 7.4, with $\Xi = Z$) also interpolates the original data.

Our first step is to derive the information $(p_{\theta,\psi} : \theta \in \mathscr{M}_0(Z), \psi \in \mathscr{D}D(Z^\theta))$ from the given data $(p_{M,\varphi} : M \in \mathscr{M}_s(X), \varphi \in \mathscr{D}(X^M))$. Precisely, we think of each $p_{M,\varphi}$ as specifying $(\varphi(D)\, p)|_M$, with p being our desired interpolant, and want to be able to compute from this information the numbers $\psi(D)\, p(\theta)$ for $\psi \in \mathscr{D}(Z^\theta)$ and $\theta \in \mathscr{M}_0(Z)$.

We fix now $\theta \in \mathscr{M}_0(Z)$ and proceed as follows: we first remove from $\mathscr{M}_s(X)$ all flats that do not contain θ. The remaining set is easily shown to coincide with $\mathscr{M}_s(X^\theta)$, i.e., the set of s-dimensional flats associated with the hyperplanes that contain θ. The information available to us at θ is of the form

$$(p_{M,\varphi}(\theta) : M \in \mathscr{M}_s(X^\theta), \varphi \in \mathscr{D}(X^M)).$$

However, since $p_{M,\varphi}$ specifies $\varphi(D)\, p$ on all of M, we thereby also know $q(D)\, \varphi(D)\, p$ on M for *any* $q \in \Pi(M)$. This means that the data supply the number $\varphi(D)\, p(\theta)$ for any

$$\varphi := \sum_{M \in \mathscr{M}_s(X^\theta)} \sum_i \varphi_i^M q_i^M \in \sum_{M \in \mathscr{M}_s(X^\theta)} \mathscr{D}(X^M)\, \Pi(M),$$

in the form

$$\varphi(D)\, p(\theta) = \sum_{M \in \mathscr{M}_s(X^\theta)} \sum_i q_i^M(D)\, p_{M,\varphi_i^M}(\theta).$$

Of course, for this to work, we must be certain that the resulting number $\varphi(D)\, p(\theta)$ is *independent* of the particular way we are writing φ as such a sum. This leads to the following.

DEFINITION 7.5. We say that the given data $(p_{M,\varphi} : M \in \mathscr{M}_s(X), \varphi \in \mathscr{D}(X^M))$ are **consistent** if, for some $Y \in \mathbb{R}^{d \times s}$ in general position with respect to X and for every $\theta \in \mathscr{M}_0(Z)$ (with $Z := X \cup Y$),

$$0 = \sum_{M \in \mathscr{M}_s(X^\theta)} \sum_i \varphi_i^M q_i^M \in \sum_{M \in \mathscr{M}_s(X^\theta)} \mathscr{D}(X^M)\, \Pi(M)$$

implies that

$$\sum_{M \in \mathcal{M}_s(X^\theta)} \sum_i q_i^M(D)\, p_{M, \varphi_i^M}(\theta) = 0.$$

Note that such consistency is demanded here only at certain finitely many points, which is very helpful, since an equivalent definition on an entire intersection of flats seems to be comparatively awkward.

It follows that the original information determines the desired information $(\psi(D)\, p(\theta) : \psi \in \mathcal{D}(Z^\theta))$ if and only if

$$\mathcal{D}(Z^\theta) \subset \sum_{M \in \mathcal{M}_s(X^\theta)} \mathcal{D}(X^M)\, \Pi(M). \tag{7.6}$$

As we now explain, it suffices to establish the above inclusion for the special case when $X^\theta = X$, and $\theta = 0$; Theorem 7.7 below establishes the inclusion for this seemingly special case. The general case of (7.6) then follows by the following reasoning.

First, since $Z^\theta \cap X = X^\theta$, and since each X^M (with $M \in \mathcal{M}_s(X^\theta)$) is a submatrix of X^θ (and not only of X), the vectors in $X \backslash X^\theta$ play no role here, hence nothing is lost in assuming $X^\theta = X$. Second, assuming $X^\theta = X$, the map

$$M \mapsto M - \theta$$

maps $\mathcal{M}_s(X)$ 1-1 onto $\mathcal{M}_{s,0}(X)$. At the same time, since all the spaces involved in (7.6) are translation-invariant, nothing is changed there if we translate them all by θ; the only change is that the index set $\mathcal{M}_s(X) = \mathcal{M}_s(X^\theta)$ of the summation is replaced by $\mathcal{M}_{s,0}(X)$.

Consequently, (7.6) will be proved as soon as we have shown the following:

THEOREM 7.7. *Let Ξ be a direction set in $\mathbb{R}^d$. Let $s \in \{1, ..., d-1\}$, and let Y be an arbitrary collection of $k \leqslant s$ vectors. Then*

$$\mathcal{D}(\Xi \cup Y) \subset \sum_{M \in \mathcal{M}_{s,0}(\Xi)} \mathcal{D}(\Xi \cap M^\perp)\, \Pi(M). \tag{7.8}$$

We note that the statement in the theorem is sharp: for example, if $\Xi \cup Y$ is in general position and Y contains exactly s vectors, then the inclusion of (7.8) already ceases to hold if we remove from the right-hand side any single summand. For, in that case, each $\mathcal{D}(\Xi \cap M^\perp)$ equals Π_0, while, for any $M \in \mathcal{M}_{s,0}(\Xi)$, $p := \ell_{\Xi \backslash M^\perp}$ is of exact degree $\#\Xi + s - d$, hence $p(D)$ fails to annihilate $\mathcal{D}(\Xi \cup Y) = \Pi_{\#\Xi + s - d}$, yet it annihilates all the summands in the right-hand side *except* $\Pi(M)$.

On the other hand, the two sides in (7.8) are never equal since the left side is contained in $\Pi_{\#\Xi+s-d}$ while the right side contains polynomials of arbitrarily high degree.

In order to prove the theorem, we observe that both sides of (7.8) are co-ideals, the right side by Proposition 6.2. The claim of the theorem follows then from the fact (about to be established) that the ideal that determines the right-hand side of (7.8) is contained in the ideal that determines the left-hand side.

As a matter of fact, $\mathscr{D}(\Xi \cup Y)$ is *defined* as the kernel of the ideal

$$\mathscr{I}(\ell_K : K \in \mathbb{K}(\Xi \cup Y)).$$

Since the space $\mathscr{D}(\Xi \cup Y)$ gets larger with increasing Y, we will assume without loss that $\# Y = s$. On the other hand, the following theorem asserts that the right-hand side of (7.8) is the kernel of the ideal $\mathscr{I}(\ell_K : K \subset \Xi, \operatorname{rank}(\Xi \backslash K) < d - s)$. Since Y has only s elements, each such K is in $\mathbb{K}(\Xi \cup Y)$. Thus, the next theorem provides a proof for (7.8).

THEOREM 7.9. *Let Ξ be a direction set in $\mathbb{R}^d$, let s be a non-negative integer $\leqslant d$, and let*

$$\mathbb{K}_s(\Xi)$$

be the collection of all $K \subset \Xi$ that are minimal with respect to the property that $\operatorname{rank}(\Xi \backslash K) < d - s$. *Then*

$$\sum_{M \in \mathscr{M}_{s,0}(\Xi)} \mathscr{D}(\Xi \cap M^\perp)\, \Pi(M) = \bigcap_{K \in \mathbb{K}_s(\Xi)} \ker \ell_K(D) =: \mathscr{D}_s(\Xi). \tag{7.10}$$

Proof. We note that, necessarily, $\operatorname{rank}(\Xi \backslash K) = d - s - 1$ for all $K \in \mathbb{K}_s(\Xi)$, and that $\mathbb{K}_0(\Xi)$ consists of all minimal elements of $\mathbb{K}(\Xi)$, hence

$$\mathscr{D}(\Xi) = \ker \mathscr{I}(\ell_K : K \in \mathbb{K}_0(\Xi)) = \mathscr{D}_0(\Xi).$$

Also, both sides of (7.10) are co-ideals (the left side by Proposition 6.2), hence their equality is equivalent to the equality of their corresponding ideals.

As a warm-up, we prove the simpler inclusion in (7.10) by showing that each of the operators $\ell_K(D)$ in (7.10) annihilates the left-hand side of (7.10). Indeed, if $K \in \mathbb{K}_s(\Xi)$ and $M \in \mathscr{M}_{s,0}(\Xi)$, then $\Xi \backslash K$ has rank $< d - s =$ rank $\Xi \cap M^\perp$, therefore $(\Xi \cap M^\perp) \backslash (K \cap M^\perp)$ has rank $< \operatorname{rank}(\Xi \cap M^\perp)$, i.e., $K \cap M^\perp \in \mathbb{K}(\Xi \cap M^\perp)$. But since, for every $q \in \Pi(M)$ and any $x \in \Xi \cap M^\perp$, $\ell_x(D)(pq) = (\ell_x(D)\, p)\, q$, this shows that $\ell_{K \cap M^\perp}(D)$ annihilates every pq with $p \in \mathscr{D}(\Xi \cap M^\perp)$ and $q \in \Pi(M)$, hence annihilates all of $\mathscr{D}(\Xi \cap M^\perp)\, \Pi(M)$; *a fortiori*, that latter space is annihilated by $\ell_K(D)$.

We now prove the converse. First, if rank $\Xi < d-s$, then $\mathcal{M}_{s,0}(\Xi) = \{\ \}$ and also $\mathbb{K}_s(\Xi) = \{\ \}$, hence both sides of (7.10) are $\{0\}$. We settle the contrary case by induction on $\#\Xi \geqslant d-s$. If $\#\Xi = d-s$, then $\mathcal{D}_s(\Xi) = \bigcap_{x \in \Xi} \ker D_x$, and hence indeed $\mathcal{D}_s(\Xi) = \Pi_0(M)$, with $M := \Xi^\perp$ the single element of $\mathcal{M}_{s,0}(\Xi)$.

Assume now that the statement of the theorem holds for Ξ, and let

$$Z := \Xi \cup \{y\}.$$

Assume that $\varphi(D)$ annihilates $\sum_{M \in \mathcal{M}_{s,0}(Z)} \mathcal{D}(Z \cap M^\perp)\, \Pi(M)$. Since this sum contains the left-hand side of (7.10), the induction hypothesis implies that $\varphi \in \mathcal{I}(\ell_K : K \in \mathbb{K}_s(\Xi))$, i.e.,

$$\varphi =: \sum_{K \in \mathbb{K}_s(\Xi)} \ell_K \psi_K,$$

where (ψ_K) are some polynomials. We need to show that $\varphi \in \mathcal{I}(\ell_U : U \in \mathbb{K}_s(Z))$. We do this by showing that

$$\ell_{K_0} \psi_{K_0} \in \mathcal{I}(\ell_U : U \in \mathbb{K}_s(Z))$$

for every $K_0 \in \mathbb{K}_s(\Xi)$.

Here is the way the proof goes. If $K_0 \in \mathbb{K}_s(Z)$, there is nothing to prove. Otherwise, $(\Xi \backslash K_0) \cup \{y\}$ is of rank $d-s$. Let $M \in \mathcal{M}_{s,0}(Z)$ be the subspace perpendicular to $(\Xi \backslash K_0) \cup \{y\}$. We will show below that $\psi_{K_0}(D)$ annihilates $\mathcal{D}((\Xi \backslash K_0) \cup \{y\})\, \Pi(M)$, hence (see the proof of Proposition 6.2) $\psi_{K_0} \in \mathcal{I}(\ell_K : K \in \mathbb{K}((\Xi \backslash K_0) \cup \{y\}))$.

The desired result follows from this since, for any $K \in \mathbb{K}((\Xi \backslash K_0) \cup \{y\})$, $Z \backslash (K_0 \cup K) = ((\Xi \backslash K_0) \cup \{y\}) \backslash K$, with the latter matrix of rank $< d-s$, therefore some submatrix of $K_0 \cup K$ lies in $\mathbb{K}_s(Z)$. Hence, $\ell_{K_0} \psi_{K_0}$ lies in the ideal generated by $\{\ell_U : U \in \mathbb{K}_s(Z)\}$ which is exactly what we had to prove.

It remains to prove that $\psi_{K_0}(D)$ annihilates $\mathcal{D}((\Xi \backslash K_0) \cup \{y\})\, \Pi(M)$. For this, we observe that

$$Z \cap M^\perp \supset (\Xi \backslash K_0) \cup \{y\} = (Z \cap M^\perp) \backslash K_0.$$

By assumption, $\sum_{K \in \mathbb{K}_s(\Xi)} \ell_K(D)\, \psi_K(D)$ annihilates $\mathcal{D}(Z \cap M^\perp)\, \Pi(M)$. In addition, there are certain terms in this sum that already annihilate $\mathcal{D}(Z \cap M^\perp)\, \Pi(M)$ (without any "aid" from other terms): if $K \in \mathbb{K}_s(\Xi)$, then $\operatorname{rank}(\Xi \backslash K) = d-s-1$, hence $(\Xi \backslash K) \not\subset M^\perp$ implies that $\operatorname{rank}((\Xi \backslash K) \cap M^\perp) < d-s-1$. Therefore, for such K, $((\Xi \backslash K) \cup \{y\}) \cap M^\perp$ is of rank $< d-s$ and, since this last matrix is $(Z \cap M^\perp) \backslash K$, we conclude that $K \cap M^\perp \in \mathbb{K}(Z \cap M^\perp)$ which means that $\ell_{K \cap M^\perp}(D)$ annihilates $\mathcal{D}(Z \cap M^\perp)$ and hence annihilates $\mathcal{D}(Z \cap M^\perp)\, \Pi(M)$. Thus, for such K, the operator $\ell_K(D)\, \psi_K(D)$

annihilates $\mathscr{D}(\mathrm{Z} \cap M^\perp)\, \Pi(M)$, and consequently, the sum of the rest of the summands annihilates this space as well.

Thus, with

$$\mathbb{K}_s^M := \{K \in \mathbb{K}_s(\Xi) : (\Xi \backslash K) \subset M^\perp\},$$

we know that

$$\sum_{K \in \mathbb{K}_s^M} (\ell_K \psi_K)(D)$$

annihilates $\mathscr{D}(\mathrm{Z} \cap M^\perp)\, \Pi(M)$. We also know that our original K_0 is in $\mathbb{K}_s^M$. Also, $(\mathrm{Z} \cap M^\perp) \backslash (K_0 \cap M^\perp) = (\mathrm{Z} \cap M^\perp) \backslash K_0 = (\Xi \backslash K_0) \cup \{y\}$. This makes the following map interesting for us.

Identifying $M^\perp$ with $\mathbb{R}^{d-s}$, we define

$$\mathscr{L}: \mathscr{D}(\mathrm{Z} \cap M^\perp) \to \mathop{\times}_{K \in \mathbb{K}_s^M} \mathscr{D}((\mathrm{Z} \cap M^\perp) \backslash (K \cap M^\perp)) :$$
$$p \mapsto (\ell_{K \cap M^\perp}(D)\, p : K \in \mathbb{K}_s^M).$$

Note that by varying K over $\mathbb{K}_s^M$, we actually vary $K \cap M^\perp$ over all minimal submatrices of $\Xi \cap M^\perp$ whose complement in $\Xi \cap M^\perp$ does not span $M^\perp$ any more. I.e., $\{K \cap M^\perp : K \in \mathbb{K}_s^M\} = \mathbb{K}_0(\Xi \cap M^\perp)$. Thus, by [DM, Theorem 3.2] (a complete proof of which can be found, e.g., in [BRS]), the map $\mathscr{L}$ is onto, and therefore, for our K_0 there exists $F \subset \mathscr{D}(\mathrm{Z} \cap M^\perp)$ such that $\ell_{K \cap M^\perp}(D)\, F = 0$ for $K \in \mathbb{K}_s^M \backslash \{K_0\}$ but $\ell_{K_0 \cap M^\perp}(D)\, F = \mathscr{D}((\mathrm{Z} \cap M^\perp) \backslash K_0)$. Thus, for $K \in \mathbb{K}_s^M \backslash \{K_0\}$,

$$\ell_K(D)\, \psi_K(D)(F\Pi(M)) = \ell_{K \backslash M^\perp}(D)\, \psi_K(D)((\ell_{K \cap M^\perp}(D)\, F)\, \Pi(M)) = \{0\},$$

and so,

$$\begin{aligned}\{0\} &= \sum \ell_K(D)\, \psi_K(D)(\mathscr{D}(\mathrm{Z} \cap M^\perp)\, \Pi(M)) \supseteq \sum \ell_K(D)\, \psi_K(D)(F\Pi(M)) \\ &= \ell_{K_0}(D)\, \psi_{K_0}(D)(F\Pi(M)) = \ell_{K_0 \backslash M^\perp}(D)\, \psi_{K_0}(D)((\ell_{K_0 \cap M^\perp}(D)\, F)\, \Pi(M)) \\ &= \psi_{K_0}(D)(\ell_{K_0 \backslash M^\perp}(D)(\mathscr{D}((\mathrm{Z} \cap M^\perp) \backslash K_0)\, \Pi(M))) \\ &= \psi_{K_0}(D)(\mathscr{D}((\mathrm{Z} \cap M^\perp) \backslash K_0)\, \Pi(M)),\end{aligned}$$

the last equality since, by Proposition 6.5, $\ell_Y(D)$ is a *surjective endo*morphism on every space of the form $F\Pi(M)$, with F a D-invariant polynomial space, provided that $\mathrm{Y} \cap M^\perp = \{\ \}$. Noting that $(\mathrm{Z} \cap M^\perp) \backslash K_0$ is exactly our old $(\Xi \backslash K_0) \cup \{y\}$, we have proved what we wanted to. ∎

Now, we finally know that the information required for solving the interpolation problem related to $\mathscr{M}_0(X \cup Y)$ is uniquely determined by the original data. Thus, by Theorem 7.4, there exists a unique polynomial in $\mathscr{P}_s(X) = \mathscr{P}(X \cup Y)$ that interpolates all the information at the points. This already implies the uniqueness of the solution to our original data.

It remains to show that this interpolant p to the pointwise data matches also the original data on $\mathscr{M}_s(X)$. Fixing $M \in \mathscr{M}_s(X)$, we begin by showing that $p|_M = p_M$, and do this by applying Theorem 7.4 to the situation on M, i.e., use, as in the simple case, the coincidence of $\mathscr{P}(Z)$ and $\mathscr{P}(Z_M)$ on M (see Lemma 5.7) to conclude that $p|_M = p_M$ from the fact that $p|_M$ "matches" p_M at each $\theta \in \mathscr{M}_0(Z) \cap M = \widetilde{\mathscr{M}}_0(Z_M)$ (see (5.12)) in the sense that

$$\varphi(D)(p|_M - p_M)(\theta) = 0, \qquad \forall \varphi \in \mathscr{D}(Z_M^\theta),$$

and, assuming consistency, this will be so provided $\mathscr{D}(Z_M^\theta) \subset \mathscr{D}(Z^\theta)$. The next lemma, applied to $\Xi = Z^\theta$, proves this containment.

LEMMA 7.11. *Let Ξ be a direction set of rank d. Let $M \in \mathscr{M}_s(\Xi)$, let P_M be the orthogonal projector onto $M - M$, and let*

$$\Xi_M := P_M(\Xi \backslash M^\perp).$$

Then,

$$\mathscr{D}(\Xi_M) \subset \mathscr{D}(\Xi). \tag{7.12}$$

Proof. Since $\mathscr{D}(\Xi)$ is the joint kernel of $\{\ell_K(D) : K \in \mathbb{K}(\Xi)\}$, it suffices to show that each $\ell_K(D)$, $K \in \mathbb{K}(\Xi)$, annihilates $\mathscr{D}(\Xi_M)$. If $K \cap (\Xi \cap M^\perp) \neq \{\ \}$, then this is obvious since in such a case $\ell_K(D)$ annihilates all of $\Pi(M)$ and in particular its subspace $\mathscr{D}(\Xi_M)$. Otherwise, $\Xi \cap M^\perp \subset \Xi \backslash K$, and therefore $P_M(\Xi \backslash K)$ cannot be of rank s (to avoid the contradictory conclusion that $\Xi \backslash K$ is of rank d, which contradicts the assumption $K \in \mathbb{K}(\Xi)$). We thus conclude that $P_M K \in \mathbb{K}(\Xi_M)$, which implies that $\ell_{P_M K}(D)\, \mathscr{D}(\Xi_M) = \{0\}$. But since $\mathscr{D}(\Xi_M) \subset \Pi(M)$, there is no difference between the action of $\ell_K(D)$ and $\ell_{P_M K}(D)$ on $\mathscr{D}(\Xi_M)$. Consequently, $\ell_K(D)\, \mathscr{D}(\Xi_M) = \{0\}$. ∎

To finish the general case, we also have to show that, for each $\varphi \in \mathscr{D}(X^M)$, $\varphi(D)\, p|_M = p_{M,\varphi}$. This we prove by induction on $j := \deg \varphi$ (having just settled the case $j = 0$), with the aid of the following theorem (which we mean to apply with $\Xi = X^\theta = Z^\theta$, hence $\Xi \cap M^\perp = X^M$).

THEOREM 7.13. *Let Ξ be a direction set of rank d. Let $M \in \mathscr{M}_s(\Xi)$, let P_M be the orthogonal projector onto $M - M$, and set*

$$\Xi_M := P_M(\Xi \backslash M^{\perp}).$$

Then, for every non-negative integer j,

$$(\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_j)\, \mathscr{D}(\Xi_M) \subset \mathscr{D}(\Xi) + (\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_{j-1})\, \Pi(M). \tag{7.14}$$

Proof. The case $j=0$ is just Lemma 7.11. The case $j>0$ is proved with the aid of differential operators, as we did in earlier proofs. Specifically, both spaces on the right-hand side of (7.14) are co-ideals (as both are homogeneous). Therefore, we are entitled to use Proposition 6.3 for the proof of (7.14). We start by identifying a generating set for the associated ideal of the second summand in the right hand side of (7.14).

LEMMA 7.15. *Let $Y \in \mathbb{R}^{d \times n}$ be a matrix of rank d. Then $\mathscr{D}(Y) \cap \Pi_{j-1}$ is the kernel of the ideal generated by the union $G_j := G_j^{\mathbb{K}} \cup G_j^{\mathbb{L}}$ of the two sets*

$$G_j^{\mathbb{K}} := \{\ell_K : K \in \mathbb{K}(Y),\ \#K \leqslant j\}, \qquad G_j^{\mathbb{L}} := \{\ell_L : L \in \mathbb{L}(Y),\ \#L = j\}.$$

Proof. It is clear that G_j generates all of $\{\ell_K : K \in \mathbb{K}(Y)\}$: if $\#K \leqslant j$, then ℓ_K appears in $G_j^{\mathbb{K}}$; otherwise, a factor of it appears in $G_j^{\mathbb{L}}$. Therefore,

$$\ker \mathscr{I}(G_j) \subset \mathscr{D}(Y).$$

On the other hand, it is clear that $\mathscr{D}(Y) \cap \Pi_{j-1}$ lies in $\ker \mathscr{I}(G_j)$, since $G_j^{\mathbb{K}}(D)$ annihilate $\mathscr{D}(Y)$, and $G_j^{\mathbb{L}}(D)$ annihilate Π_{j-1}.

Thus, we only need to show that $\ker \mathscr{I}(G_j) \subset \Pi_{j-1}$, or, in other words, that G_j generates all of Π_j^0. For this, observe that $\mathscr{I}_{\mathscr{D}(Y)} + \mathscr{P}(Y) = \Pi$. Since both summands are homogeneous, $(\mathscr{I}_{\mathscr{D}(Y)} \cap \Pi_j^0) + (\mathscr{P}(Y) \cap \Pi_j^0) = \Pi_j^0$. But, $G_j^{\mathbb{K}}$ generates $\mathscr{I}_{\mathscr{D}(Y)} \cap \Pi_j^0$ and $G_j^{\mathbb{L}}$ spans $\mathscr{P}(Y) \cap \Pi_j^0$, whence the desired conclusion. ∎

COROLLARY 7.16. *The ideal whose kernel is $(\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_{j-1})\, \Pi(M)$ is generated by $G_j := G_j^{\mathbb{K}} + G_j^{\mathbb{L}}$, with*

$$G_j^{\mathbb{K}} := \{\ell_K : K \in \mathbb{K}_0(\Xi \cap M^{\perp}),\ \#K \leqslant j\} \qquad \text{and}$$

$$G_j^{\mathbb{L}} := \{\ell_L : L \in \mathbb{L}(\Xi \cap M^{\perp}),\ \#L = j\}.$$

Combining this corollary with Proposition 6.3, we see that our Theorem 7.13 follows from the following claim:

PROPOSITION 7.17. *Let G_j be as in the preceding corollary. Then, for every $p \in (\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_j)\, \mathscr{D}(\Xi_M)$, there exists $f \in \mathscr{D}(\Xi)$ such that*

$$\ell_{\mathrm{Z}}(D)(p-f)=0, \qquad \forall \ell_{\mathrm{Z}} \in G_j.$$

Proof. Let

$$P_{\Xi}$$

be the projector of Π onto $\mathscr{D}(\Xi)$ with respect to $\mathscr{P}(\Xi)$. Namely,

$$\varphi(D)(p-P_{\Xi}p)(0)=0, \qquad \forall \varphi \in \mathscr{P}(\Xi), \forall p \in \Pi.$$

This projector exists (and is unique) because of the duality between $\mathscr{D}(\Xi)$ and $\mathscr{P}(\Xi)$. It is proved in [DR] (cf. Section 6 there) that for every $Y \subset \Xi$ we have

$$\ell_Y(D) P_{\Xi} = P_{\Xi \backslash Y} \ell_Y(D).$$

Now let $p \in (\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_j)\, \mathscr{D}(\Xi_M)$ and choose $q := P_{\Xi} p \in \mathscr{D}(\Xi)$. Let $\ell_{\mathrm{Z}} \in G_j$; then, in particular, $\mathrm{Z} \subset \Xi \cap M^{\perp} \subset \Xi$, and hence, by the above,

$$\ell_{\mathrm{Z}}(D)(p-q)=\ell_{\mathrm{Z}}(D)p - P_{\Xi\backslash \mathrm{Z}}(\ell_{\mathrm{Z}}(D)p). \tag{7.18}$$

We will show now that, whatever $\ell_{\mathrm{Z}} \in G_j$ was chosen, $\ell_{\mathrm{Z}}(D)\, p \in \mathscr{D}(\Xi \backslash \mathrm{Z})$. Since $P_{\Xi \backslash \mathrm{Z}}$ is the identity on $\mathscr{D}(\Xi \backslash \mathrm{Z})$, (7.18) will then imply that $\ell_{\mathrm{Z}}(D)(p-f)=0$.

Whatever Z we did choose, $\mathrm{Z} \subset M^{\perp}$, and hence

$$\ell_{\mathrm{Z}}(D)(\mathscr{D}(\Xi \cap M^{\perp})\, \Pi(M)) = (\ell_{\mathrm{Z}}(D)\, \mathscr{D}(\Xi \cap M^{\perp}))\, \Pi(M).$$

If $\mathrm{Z} \in \mathbb{K}(\Xi \cap M^{\perp})$, then $\ell_{\mathrm{Z}}(D)p = 0 \in \mathscr{D}(\Xi \backslash \mathrm{Z})$. Otherwise, $\ell_{\mathrm{Z}} \in G_j^{\mathbb{L}}$ and hence $\#\mathrm{Z}=j$, and therefore $\ell_{\mathrm{Z}}(D)(\mathscr{D}(\Xi \cap M^{\perp}) \cap \Pi_j) \subset \Pi_0$. Consequently, $\ell_{\mathrm{Z}}(D)p \in \mathscr{D}(\Xi_M) \subset \mathscr{D}(\Xi \backslash M^{\perp}) \subset \mathscr{D}(\Xi \backslash \mathrm{Z})$, the middle inclusion by virtue of the proven $j=0$ case of the theorem. This completes the proof of the present proposition, and thereby completes the proof of the whole Theorem 7.13. ∎

With this, we have almost proved the following main result of this paper.

THEOREM 7.19. *Let X be a direction set in $\mathbb{R}^d$, and let $(H_x : x \in X)$ be a corresponding sequence of hyperplanes, each perpendicular to its associated x. For a fixed $s \in \{0, ..., d-1\}$, let there be given consistent (see Definition 7.5) and X-compatible (see Definition 7.3) data $(p_{M,\varphi} : M \in \mathscr{M}_s(X), \varphi \in \mathscr{D}(X^M))$. Then, there exists exactly one $p \in \mathscr{P}_s(X)$ that satisfies*

$$(\varphi(D)\, p)|_M = p_{M,\varphi}, \qquad \forall M \in \mathscr{M}_s(X), \quad \forall \varphi \in \mathscr{D}(X^M).$$

Proof. The only thing still not proved is the claim that the (unique) interpolant $p \in \mathscr{P}(X \cup Y)$ to the data

$$(p_{\theta,\psi} : \theta \in \mathscr{M}_0(X \cup Y), \psi \in \mathscr{D}((X \cup Y)^\theta))$$

(derived with the aid of Theorem 7.7, as explained before) satisfies

$$\varphi(D)p|_M = p_{M,\varphi}$$

for all $\varphi \in \mathscr{D}(X^M)$ and all $M \in \mathscr{M}_s(X)$. We prove this by induction on $j := \deg \varphi$, the case $j = 0$ having already been settled.

Consider $\varphi \in \mathscr{D}(X^M)$ of degree j. Since $p \in \mathscr{P}_s(X)$, so is $\varphi(D)p$, hence $\varphi(D)p|_M \in \mathscr{P}_s(X)|_M$, while $p_{M,\varphi} \in \mathscr{P}_s(X)|_M$ by the assumed X-compatibility of the data. By Theorem 7.4 and Lemma 5.7, it is therefore sufficient to prove that

$$\psi(D)(\varphi(D)p|_M - p_{M,\varphi})(\theta) = 0, \qquad \forall \theta \in \mathscr{M}_0(Z) \cap M, \qquad \forall \psi \in \mathscr{D}(Z_M^\theta).$$

By Theorem 7.13, any such $\psi\varphi$ is expressible as a finite sum

$$\psi\varphi = \psi_0 + \sum_i \varphi_i \psi_i,$$

with $\psi_0 \in \mathscr{D}(Z^\theta)$ and $\varphi_i \in \mathscr{D}(Z^M) \cap \Pi_{j-1}$, $\psi_i \in \Pi(M)$, all i. By Theorem 7.7,

$$\psi_0 = \sum_{N \in \mathscr{M}_{s,0}(Z^\theta)} \sum_i \varphi_i^N \psi_i^N$$

for some $\varphi_i^N \in \mathscr{D}(Z \cap N^\perp)$, $\psi_i^N \in \Pi(N)$, and, by construction of p,

$$\psi_0(D)p(\theta) = \sum_{N \in \mathscr{M}_{s,0}(Z^\theta)} \sum_i \psi_i^N(D) p_{N,\varphi_i^N}(\theta).$$

Therefore,

$$\begin{aligned}\psi(D)\, \varphi(D)p(\theta) &= \left(\psi_0(D) + \sum_i \varphi_i(D)\, \psi_i(D)\right) p(\theta)\\ &= \sum_{N \in \mathscr{M}_{s,0}(Z^\theta)} \sum_i \psi_i^N(D) p_{N,\varphi_i^N}(\theta) + \sum_i \psi_i(D) p_{M,\varphi_i}(\theta)\\ &= \psi(D) p_{M,\varphi}(\theta),\end{aligned}$$

the second equality by induction hypothesis and the last equality by the assumed consistency of the data. ∎

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